Math 121 3.6-2nd Related Rates

Objectives

1) Differentiate an equation with respect to time to get a related rates equation.
2) Solve related rates problems.

Recall: $\frac{dy}{dx}$ = rate of change of y with respect to x.

This derivative tells how y changes when x changes

In this section, we find $\frac{dy}{dt}$ or $\frac{dx}{dt}$ or $\frac{dV}{dt}$ or $\frac{dr}{dt}$.

These are all derivatives with respect to t, time, and are used when all variables in a problem change with time.

Ex. If we blow up a spherical balloon,

the rate of change of the radius r is $\frac{dr}{dt}$.

The rate of change of the volume V is $\frac{dV}{dt}$.

Ex. If pollution depends on the size of the population and both change over time

the rate of change of pollution S is $\frac{dS}{dt}$

the rate of change of population x is $\frac{dx}{dt}$.

Related rates is an application of implicit differentiation

$\Rightarrow$ whenever we take a derivative of <u>any</u> variable,

we multiply by a chain rule

Find derivatives.

① $\frac{d}{dx}(x^3)$

$= \boxed{3x^2}$

② $\frac{d}{dx}(y^3)$

$= \boxed{3y^2 \cdot \frac{dy}{dx}}$

③ $\frac{d}{dt}(x^3)$

$= \boxed{3x^2 \frac{dx}{dt}}$

An <u>equation</u> tells how two variables are related.

A <u>related rates</u> equation tells how the rates of change of the variables in that equation are related to each other.

Find related rates equations.

(4) $x^5 - y^3 = 1$

Differentiate all terms, both sides, with respect to time, t.

$$\boxed{5x^4 \frac{dx}{dt} - 3y^2 \cdot \frac{dy}{dt} = 0.}$$

(5) $xy^2 = 96$

product rule

$$\frac{d}{dt}(x) \cdot y^2 + \frac{d}{dt}(y^2) \cdot x = \frac{d}{dt}(96)$$

$$\frac{dx}{dt} y^2 + 2y \frac{dy}{dt} \cdot x = 0$$

$$\boxed{y^2 \frac{dx}{dt} + 2xy \frac{dy}{dx} = 0}$$

(6) $2x^3 - 5xy = 14$

$$6x^2 \frac{dx}{dt} - 5\left[x \cdot \frac{dy}{dt} + \frac{dx}{dt} \cdot y\right] = 0$$

$$\boxed{6x^2 \frac{dx}{dt} - 5x \frac{dy}{dt} - 5y \frac{dx}{dt} = 0}$$

(7) $x^3 - xy = y^3$

$$3x^2 \frac{dx}{dt} - \left[x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}\right] = 3y^2 \frac{dy}{dt}$$

$$\boxed{3x^2 \frac{dx}{dt} - x \frac{dy}{dt} - y \frac{dx}{dt} = 3y^2 \frac{dy}{dt}}$$

To solve a related rate problem

1) Determine which quantities are changing with time.
   Identify variables.
2) Find an equation that relates these variables.
3) Differentiate implicitly, with respect to time t, using chain rule for derivatives of any variable that changes with time.
   This new equation is the related rates equation.
4) Identify gives values for variables and rates of change of those variables.
   Identify what derivative is requested by the question.
5) Isolate the desired derivative to answer the question.

CAUTION: Do not substitute any values before differentiating.
(The derivative of a constant is 0 $\Rightarrow$ but these concepts are changing, not constant!)

(8) While blowing up a spherical balloon, the rate of change of the radius is 2 cm per second. Find the rate of change of the volume when the radius is 8 cm.

quantities: r = radius
V = volume

equation: volume of a sphere $V = \frac{4}{3}\pi r^3$

differentiate: $\frac{dV}{dt} = \underbrace{\frac{4}{3}\pi}_{\text{constant multiples}} \cdot \frac{d}{dt}(r^3)$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3 \cdot r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$ ← This is the related rates equation.

Given: $\frac{dr}{dt} = 2\frac{cm}{sec}$ $\quad r = 8cm \quad$ find $\frac{dV}{dt}$.

$$\frac{dV}{dt} = 4\pi(8)^2 \cdot 2 = \boxed{512\pi \frac{cm^3}{sec}}$$

units of volume

units of time

Hint: Units can help if you substitute in, you'll find needed units.

In example 8: $r = 8\,cm$

$$\frac{dr}{dt} = \frac{2\,cm}{sec}$$

$$\frac{dV}{dt} = 4\pi(8\,cm)^2 \cdot \frac{2\,cm}{sec}$$

$$= 4 \cdot \pi \cdot 8 \cdot cm \cdot 8 \cdot cm \cdot 2 \cdot \frac{cm}{sec}$$

$$= 4\pi \cdot 8 \cdot 8 \cdot 2 \cdot \frac{cm \cdot cm \cdot cm}{sec}$$

$$= \boxed{512\pi \frac{cm^3}{sec}}$$

(9) The balloon got away before we tied it off. The rate of change of the volume is $768\pi\,cm^3/sec$. Find the rate of change of the radius when the radius is 6 cm.

$$\left.\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt} \end{aligned}\right\} \text{ as before}$$

$$\frac{dV}{dt} = -768\pi\,cm^3/sec$$

negative because the volume is decreasing!

$r = 6\,cm$

$$-768\pi \frac{cm^3}{sec} = 4\pi(6\,cm)^2 \cdot \frac{dr}{dt}$$

$$-768\pi \frac{cm^3}{sec} = 4\pi \cdot 36\,cm^2 \cdot \frac{dr}{dt}$$

$$-768\pi \frac{cm^3}{sec} = 144\pi\,cm^2 \frac{dr}{dt}$$

$$\frac{-768\pi\,cm^3}{144\pi\,cm^2 \cdot sec} = \frac{dr}{dt}$$

$$\boxed{\frac{-16}{3}\frac{cm}{sec} = \frac{dr}{dt}}$$

negative because radius is decreasing as the balloon deflates

(10) A study of urban sulfur dioxide pollution emissions in a city shows that pollution S in tons is given by

$$S = 2 + 20x + 0.1x^2 \quad \text{tons}$$

where x is the population in thousands.

The population of the city t years from now is expected to be

$$x = 800 + 20\sqrt{t} \quad \text{thousand people.}$$

Find how rapidly the sulfur dioxide pollution will be increasing 4 years from now.

Differentiating both equations with respect to time t:

$$\frac{dS}{dt} = 0 + 20 \cdot \frac{dx}{dt} + 0.2x \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = 0 + 20 \cdot \frac{1}{2} t^{-1/2}$$

note $\frac{dt}{dt} = 1$!

simplify: $\frac{dx}{dt} = \frac{10}{\sqrt{t}}$

subst $\frac{dx}{dt}$ into $\frac{dS}{dt}$ equation; subst $x = 800 + 20\sqrt{t}$ also

$$\frac{dS}{dt} = 20 \cdot \frac{10}{\sqrt{t}} + 0.2(800 + 20\sqrt{t}) \cdot \frac{10}{\sqrt{t}}$$

$$= \frac{200}{\sqrt{t}} + \frac{2}{\sqrt{t}}(800 + 20\sqrt{t})$$

$$= \frac{200}{\sqrt{t}} + \frac{1600}{\sqrt{t}} + 40$$

$$\frac{dS}{dt} = \frac{1800}{\sqrt{t}} + 40$$

Subst t=4

$$\frac{dS}{dt} = \frac{1800}{\sqrt{4}} + 40$$

$$= \boxed{940 \text{ tons/yr}}$$

Method 2: Subst $x \to S$ before differentiating.

$$S = 2 + 20\left(800 + 20\sqrt{t}\right) + 0.1\left(800 + 20\sqrt{t}\right)^2$$

$$\frac{dS}{dt} = 0 + 20\left(0 + 20\cdot\tfrac{1}{2}t^{-1/2}\right) + (0.1)(2)(800 + 20\sqrt{t})\left(0 + 20\cdot\tfrac{1}{2}t^{-1/2}\right)$$

$$= 20\cdot 10\cdot t^{-1/2} + 0.2\left(10t^{-1/2}\right)\left(800 + 20t^{1/2}\right)$$

$$= 200t^{-1/2} + 2t^{-1/2}\left(800 + 20t^{1/2}\right)$$

$$= 200t^{-1/2} + 1600t^{-1/2} + 40$$

$$= 1800t^{-1/2} + 40 \qquad \text{same as before}$$

$$= \frac{1800}{\sqrt{t}} + 40.$$

Method 3: Use the $\frac{dS}{dt}$ and $\frac{dx}{dt}$ without simplifying.

$$\frac{dx}{dt} = \frac{10}{\sqrt{t}}$$

Subst $t=4$ into $\frac{dx}{dt}$:

$$\frac{dx}{dt} = \frac{10}{\sqrt{4}} = \frac{10}{2} = 5$$

Subst $t=4$ into x:

$$x = 800 + 20\sqrt{4} = 800 + 20(2) = 840$$

Subst x and $\frac{dx}{dt}$ into $\frac{dS}{dt}$:

$$\frac{dS}{dt} = 20\cdot(5) + 0.2(840)\cdot(5)$$

$$= 100 + 840$$

$$= \boxed{940 \text{ tons/yr.}}$$